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The topographic control of planetary-scale flow

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We develop a theory to describe the topographic control of planetary-scale flows resulting from the variation of the Earth's rotation with latitude. We show that on passing over topography, an inertial, zonal current on an equatorial β-plane may pass through a control at which the flow changes from a subcritical to a supercritical solution branch. Downstream of this control, a transition back to the subcritical solution branch may occur, for example, by the generation of planetary eddies or radiating Rossby waves. We calculate the energy dissipated across such a transition and discuss the relevance of this theory for a number of atmospheric and oceanic phenomena. We also show that this phenomenon is analogous to the hydraulic control of a non-rotating, stratified flow passing through a channel of variable width.

1. Introduction

There are many examples of eastward flowing zonal currents in the atmosphere, the ocean and on the planets, including the equatorial counter-current in the Pacific ocean (Cromwell, Montgomery & Stroup 1954), the zonal currents on the planets Jupiter, Saturn and Venus, and the atmospheric jetstream. These flows occur in relatively thin shells, and many useful insights have been gained by approximating the motion as occurring on a barotropic β-plane (Pedlosky 1987; Gill 1982; Armi 1989). In many cases the dominant dynamical balance is between inertia and the variation of the Coriolis force with latitude (the β-effect). Therefore, we focus attention upon inviscid, inertial, barotropic flows on a β-plane. In order to study some of the underlying dynamical features of such flows, we have adopted some techniques from the theory of stratified hydraulics (Benjamin 1981).

Rossby (1949) initiated the hydraulic theory of planetary-scale flows by showing that for certain values of the energy flux there may be two possible current widths, if one assumes the structure of a zonal flow. Armi (1989) developed this work, arguing that if the along-current pressure field changes, it is possible that a subsonic current may pass through a control point and become supersonic. However, in both the original theory of Rossby (1949) and that of Armi (1989) the shape of the current was assumed rather than deduced from the equations of motion; also, their theories did not include a mechanism, such as topography, by which the current may evolve from one state to the other. These studies were motivated by an analogy with the hydraulics of a non-rotating barotropic fluid; however, motion on a β-plane is in fact analogous to a special class of stratified flows (Ball 1954).

In this paper, we calculate some exact inertial solutions which describe the motion of zonal currents on an equatorial β-plane flowing over topography. We construct these solutions by assuming that the velocity profile upstream and downstream of the topography is the same but we allow the flow downstream to be of a different width and intensity. This technique for calculating steady, nonlinear zonal flows on an inertial β-plane is equivalent to that introduced by Benjamin (1981) in the context of non-rotating, stratified channel flow. Using our solutions, we demonstrate how a zonal current may be controlled by variations in bottom topography; as the flow passes over the topography, the flow may change solution branch, being subcritical upstream and supercritical downstream of the topography. Although the two solutions have the same shape, the slow subcritical flow spans a wide latitude, while the fast supercritical flow downstream is much 'narrower'. Downstream of the topography, the supercritical flow may subsequently pass through a transition back to the subcritical solution. This transition, which corresponds to a widening of the current, may be manifested by eddies or the radiation of planetary waves; we calculate the energy dissipated in this transition. In the usual context of hydraulics in a stratified flow, the terms subcritical and supercritical generally refer to the speed of the current relative to that of internal gravity waves. By direct analogy, in the present context the criticality is in relation to long Rossby waves.

We also show that at midlatitudes, for a given current transport upstream of the topography, there are two possible current widths downstream of the topography; however, in this case, the Coriolis force produced by the background rotation results in a different self-similar velocity profile for each solution branch. Variations in the coastline latitude may also induce multiple flow solutions. We show that for certain transports of the zonal, planetary-scale flow there are two families of solutions in which the flow upstream and downstream of a change in boundary latitude have the same velocity profile; these solutions are similar to the two classes of solutions which describe midlatitude zonal flows. In §6, we briefly discuss the relevance of this phenomenon in the interpretation of planetary-scale flows.

Our fully nonlinear solutions are quite different from those of Charney & DeVore (1979) and Hart (1979), who found multiple solutions in the problem of forced zonal flow over shallow ridges. Also, the present work differs fundamentally from the studies of Sambuco & Whitehead (1976), Gill (1977) and Dalziel (1990) in which the hydraulics of currents on an f-plane were examined. Their theories were an extension of the theory of non-rotating hydraulics through the inclusion of an extra pressure force due to the rotation. We do not include any gravitational forces; all the dynamics that we discuss emerge from the variation of the Coriolis acceleration with latitude which has an effect analogous to the gravitational force acting on a non-rotating stratified fluid. The present work also differs from that of Luyten & Stommel (1985) and Ou & DeReuter (1987) who considered the effects of baroclinicity as well as the β-effect. We model the upper and lower boundaries of the zonal flow as rigid, with the depth of the flow affecting only the conservation of mass.

In the Appendix we show that the zonal flows described by our hydraulic theory are analogous to a subclass of stratified channel flows in which the density is a linear function of the streamfunction. The presence of a north–south oriented ridge on a β-plane is equivalent to a constriction which decreases the width of the channel, while a variation in the latitude of the coastline is equivalent to a change in the depth of the channel. This analogy between stratified and rotating flow is different from the classical analogy described by Veronis (1970) in which the background rotation was assumed to be constant; we consider two-dimensional zonal flows on a β-plane and compare them with two-dimensional, horizontal flows of a stratified current.

2. The zonal current

2.1. *Conserved quantities*

We consider zonal, inviscid motions on a barotropic β-plane. The steady, two-dimensional equations of motion on a barotropic β-plane may be written as (Pedlosky 1987; Gill 1982)

$$uu_x + vu_y - (f_0 + \beta y)\,v = -P_x \tag{2.1a}$$

and

$$uv_x + vv_y + (f_0 + \beta y)\,u = -P_y, \tag{2.1b}$$

where P is the sum of the pressure field and any external, conservative forces, f_0 is the background rotation; we have assumed that the fixed fluid density on the barotropic β-plane is unity for simplicity of notation. Conservation of mass allows the introduction of a transport streamfunction ψ where

$$H(x)\,[u,v] = [-\psi_y, \psi_x], \tag{2.2}$$

and $H(x)$ is the depth of the fluid (in this paper we assume the depth varies only in the zonal direction, for simplicity). The notation $[a,b]$ represents a vector in the (x, y)-plane. Equations (2.1a,b) may be added to give

$$(\boldsymbol{u}\cdot\boldsymbol{\nabla})\,\boldsymbol{u} - \frac{f_0 + \beta y}{H}\boldsymbol{\nabla}\psi = -\boldsymbol{\nabla}P. \tag{2.3}$$

Since $(\boldsymbol{u}\cdot\boldsymbol{\nabla}\psi) = 0$, by taking the vector product of $\boldsymbol{u}$ with (2.3), we deduce that the quantity $\frac{1}{2}\boldsymbol{u}^2 + P$ is conserved following the flow, and this is the β-plane version of the Bernoulli constant, as noted by Ball (1954). Equation (2.3) suggests that in a region of constant depth, the Coriolis force resulting from the mean rotation of the flow, f_0, contributes the quantity $-f_0\psi/H$ to the pressure field. By including this term in the pressure field, Sambuco & Whitehead (1976) and Gill (1977) developed hydraulic theories for the motion of fluid in a channel on an f-plane subject to both gravitational and rotational forces. In the present problem we have imposed a rigid lid on the β-plane and therefore ignore the gravitational forces; the only effect of varying the topography is to change the speed and latitudinal span of the flow.

The Bernoulli function may be used to derive a number of conserved quantities. First, if the zonal current varies continuously and smoothly, with no jumps or sudden transitions, then the energy flux, G, defined as

$$G(x) = \int_0^{-H_0\Phi}\left(\frac{u^2}{2} + P\right)\mathrm{d}\psi = H(x)\int_0^L u\left(\frac{u^2}{2} + P\right)\mathrm{d}y \tag{2.4}$$

is conserved with the flow, where $H_0\Phi$ is the total transport. That the energy flux (2.4) is a constant follows by integrating the Bernoulli constant across streamlines.

Secondly, we let $\boldsymbol{A} = (0, 0, B)$, where B is the Bernoulli function, and note that

$$\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\boldsymbol{A}) = 0. \tag{2.5}$$

By evaluating $\boldsymbol{\nabla}\times\boldsymbol{A}$ using (2.1) and substituting in the conservation of mass (2.2), one may re-express equation (2.5) in the form

$$\boldsymbol{u}\cdot\boldsymbol{\nabla}\left(\frac{w + f_0 + \beta y}{H}\right) = 0, \tag{2.6}$$

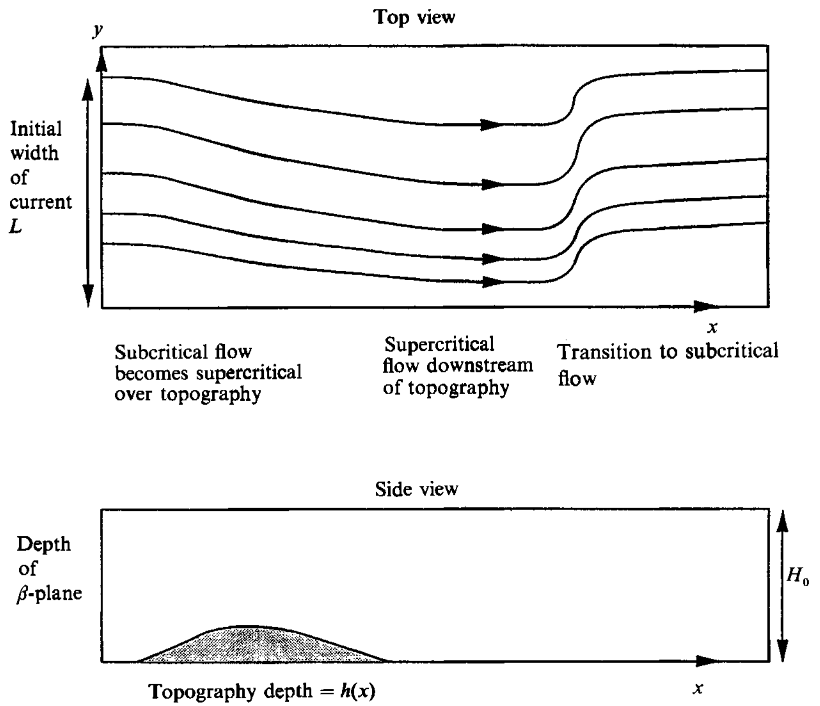

FIGURE 1. The geometry of the zonal, planetary scale flow over topography.

where $w = v_x - u_y$. This equation expresses the conservation of the potential vorticity $(w+f_0+\beta y)/H$ along streamlines.

2.2. *The boundary conditions*

We consider inertial currents propagating along a free-slip zonal boundary at $y = 0$, for example a coastline or a mountain range which runs east–west (figure 1). We restrict attention to flows of finite width which lie in the region $0 < y < L(x)$. In a quiescent environment, the outer, free streamline of the current satisfies $\psi_y(L) = 0$ and since the total eastward transport in the flow is fixed, $\psi(0)-\psi(L) = H_0 \Phi > 0$. For convenience we choose $\psi(0) = 0$.

3. Topographic control on a β-plane

Using the constraints described in §2, we investigate the motion of such zonal flows as they travel through a region of variable depth; in particular, we wish to determine whether the current can evolve smoothly from one velocity profile to another and thereby be hydraulically controlled by the region of variable depth. Since a uniform finite current will not satisfy the boundary conditions, one approach is to seek self-similar flow solutions, as is the practice in stratified channel hydraulics (Wood 1968; Benjamin 1981). We may derive such solutions using the conservation of potential vorticity along streamlines; this is analogous to, but simpler than, using the conservation of Bernoulli function along streamlines, as is usually the practice in stratified hydraulics.

We consider the bottom topography to be uniform with latitude but to vary in the zonal direction. The topography may be of arbitrary shape as long as it varies sufficiently slowly with latitude that the flow is everywhere approximately parallel. We define the depth of fluid to be $H(x) = \mu(x) H_0$, where x is the zonal position, and $H \to H_0$ as $x \to -\infty$, as sketched in figure 1.

Far upstream of the topography, in the region of constant depth H_0, we denote the streamfunction by $\Psi(y)$. Therefore, if δl represents the longitudinal distance between two streamlines, at some zonal position x, through which the net transport is $\delta\Psi$, then we may write

$$\mu(x) H_0 u(x,y)\, \delta l = -\delta\Psi, \tag{3.1}$$

giving the result that

$$\frac{\partial \Psi}{\partial y} = -u\mu H_0 \frac{\partial l}{\partial y}. \tag{3.2}$$

$l(y;x)$ represents the latitude of the transport streamline $\Psi(y)$ (which had latitude y far upstream) at the zonal position x.

If the mass transport streamline ψ has latitude y_1 and y_2 at two different zonal locations x_1 and x_2 of depth μ_1 and μ_2, then the conservation of potential vorticity along streamlines, (2.6), may be written

$$\mu_2^2[\psi_{yy}(x_1, y_1) + \mu_1 H_0(\beta y_1 + f_0)] = \mu_1^2[\psi_{yy}(x_2, y_2) + \mu_2 H_0(\beta y_2 + f_0)] \tag{3.3}$$

assuming that μ changes slowly with x. If we set $y_2 = \lambda y_1$, $\mu_1 = 1$, $\mu_2 = \mu$ and we seek a self-similar velocity profile, Ψ, defined such that

$$\Psi(y) = \psi(x_1, y) \tag{3.4}$$

and

$$\psi(x_2, \lambda y) = \Psi(y), \tag{3.5}$$

then

$$\psi_{yy}(x_1, y) = \lambda^2 \psi_{yy}(x_2, \lambda y) = \Psi_{yy}(y). \tag{3.6}$$

Combining (3.3)–(3.6) it follows that

$$\Psi_{yy}(1-\mu^2\lambda^2) = H_0 \lambda^2 \mu[\beta y(\mu - \lambda) + f_0(\mu - 1)]. \tag{3.7}$$

The similarity solution satisfying this equation and the boundary conditions is

$$\Psi(y) = \frac{H_0 \lambda^2 \mu}{6(1-\lambda^2\mu^2)}[\beta(\mu-\lambda)(y^3 - 3yL^2) + 3f_0(\mu-1)(y^2 - 2yL)], \tag{3.8}$$

where L is the outer edge of the current at the zonal position x_1. In order that the total transport remains fixed with value $H_0 \Phi$, we require

$$\Phi = \frac{\lambda^2 \mu L^2}{6(1-\lambda^2\mu^2)}[2\beta L(\mu - \lambda) + 3f_0(\mu - 1)]. \tag{3.9}$$

To simplify this expression we introduce two dimensionless parameters. We define the variable

$$R_\beta = 3\Phi/\beta L^3 \tag{3.10}$$

to be an approximate Rossby number for the upstream flow since Φ/L^2 is a measure of the dynamic or relative vorticity in the current while βL is a scale for the change in the planetary vorticity across the flow. We also define the variable

$$R_f = 3f_0/2\beta L \tag{3.11}$$

which is a measure of the background value of the Earth's rotation in comparison with the variation of the Earth's rotation across the current; in planetary-scale flows on Earth this is typically in the range 0–10, depending upon the latitude.

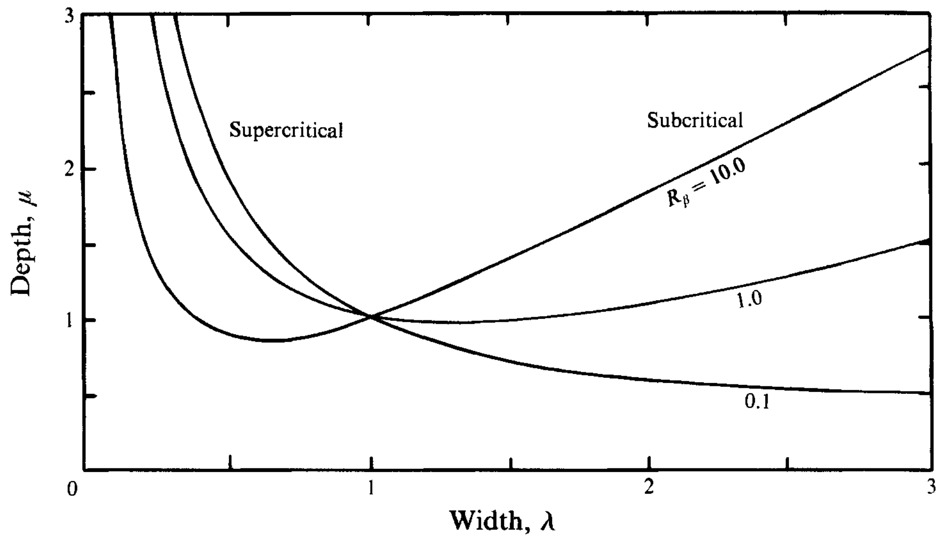

FIGURE 2. The dependence of the dimensionless current width λ upon the dimensionless depth μ for $R_\beta = 0.1$, 1.0 and 10 with $R_f = 0$.

Substituting in these definitions for R_β and R_f, we may now rewrite the cubic (3.9) for λ in the form

$$\mu\lambda^3 - [\mu(1+R_\beta)+R_f(\mu-1)]\mu\lambda^2+R_\beta = 0. \tag{3.12}$$

This cubic may now be used to relate the width, λ, of the current downstream of the topography to that upstream if the velocity profile has the same shape and that the depth downstream is a fraction μ of that upstream. If the depth of the fluid returns to H_0 downstream of the topography, then the term representing the background rotation, $R_f(\mu-1)$ becomes zero; in this case, even though the velocity profile of the current changes as it flows over the topography, downstream of the topography, the width of the current, λ, is independent of the background rotation.

3.1. *Equatorial zonal currents*

In the particular case of an equatorial zonal current, in which $f_0 = 0$, the self-similar velocity profile (3.8) applies at all points over the topographic bump, as well as at points far upstream and downstream of the topography. The solution of (3.12) gives the width of the current $\lambda(x)$ at each zonal location in terms of the depth $\mu(x)$. Therefore (3.8) represents a self-similar flow at all positions over the topography; in this solution, even if the depth, and therefore width, of the current change, the velocity profile remains the same.

We have plotted the roots of this cubic (3.12) in figure 2 for $R_\beta = 0.1$, 1 and 10 and $R_f = 0$. The variable λ, which represents the latitudinal span of the current relative to its value upstream, has two real positive roots when

$$4(1+R_\beta)^3\mu^4 > 27R_\beta. \tag{3.13}$$

These two solution branches converge when $\mu = \mu_c$ where

$$4(1+R_\beta)^3\mu_c^4 = 27R_\beta \tag{3.14}$$

at which point

$$\lambda = \lambda_c = \tfrac{2}{3}\mu(1+R_\beta). \tag{3.15}$$

As μ increases beyond the value $\mu = \mu_c$ one of the solutions λ_{sup}, representing the fast supercritical flow, approaches zero, while the other solution λ_{sub}, representing the slower subcritical flow, tends to the value $(1+R_\beta)\mu$.

If the depth of the fluid at the topographic bump decreases to the critical value of $\mu = \mu_c$, where $\mu_c = (27R_\beta/4(1+R_\beta)^3)^{\frac{1}{3}}$, then the flow may change smoothly from the subcritical solution branch, characterized by a slow wide flow to the supercritical solution branch, characterized by the faster, narrow flow (see figure 2). The flow solution which changes solution branch on passing over the topography is in fact the 'choked flow'; it is exactly analogous to the flow of a stratified fluid from a reservoir through a channel which initially contracts and subsequently expands into a second reservoir, as discussed by Benjamin (1981). If the topography is so large that μ becomes smaller than μ_c, then the self-similar solution breaks down.

If the flow propagates supercritically downstream of the hydraulic control point, and the fluid depth increases, the zonal current becomes increasingly narrower. At some point downstream, therefore, a transition in the current back to the wide subcritical flow may occur. Such a transition would result from nonlinear Rossby waves which can propagate westwards towards the topography from far downstream, in an analogous fashion to nonlinear gravity waves which propagate upstream to cause a hydraulic jump in classical hydraulics (Turner 1979). In §4, we estimate the energy lost across such a jump, and infer a possible scaling for planetary eddies.

3.2. *Non-equatorial zonal flows*

In the general case $R_f \neq 0$, solutions of the form (3.8) represent self-similar flows upstream and downstream of a topographic obstacle. However, although the flow upstream and downstream has the same velocity profile, as the flow passes over the topography, it has a different profile. This is because the potential vorticity, f_0/H, associated with the background rotation f_0, changes as the depth H changes. In the case that the depth returns to its original value $\mu = 1$ downstream both the original profile and the self-similar solution branch downstream are the same as for an equatorial β-plane (§3.1).

It is interesting to note that the solution (3.8) also identifies that self-similar solutions exist upstream and downstream of a region of variable depth, even if the depth upstream and downstream are different. In figure 3, we have plotted the dependence of λ upon μ for $R_\beta = 0.1$ and $R_f = 0.1$, 1 and 10. This figure shows that the existence of multiple solutions for λ for a given downstream depth, μ, depends upon the latitude. The minimum value of μ which yields two solution branches for the width of the flow downstream increases with R_f. However, there are always two solution branches in the neighbourhood of $\mu = 1$, since when $\mu = 1$ the latitude of the current has no effect upon the width. An important difference between these solutions and those described for equatorial zonal flows is that now the velocity profile is different on each solution branch. If the flow lies on the narrow supercritical solution branch, then after passing over the topography, the self-similar current retains its velocity profile, but changes width according to the part of the curve with negative slope in figure 3. If the flow is on the wider subcritical solution branch, then the self-similar current again retains its velocity profile after passing over the topography, but now the width changes according to the part of the curve with positive slope shown in figure 3. In each of these cases, the shape of the velocity profile is a function of the depth downstream of the topography. The two solution branches coincide when the depth μ satisfies the relation

$$4\mu(\mu(1+R_\beta)+R_f(\mu-1))^3 = 27R_\beta \qquad (3.16)$$

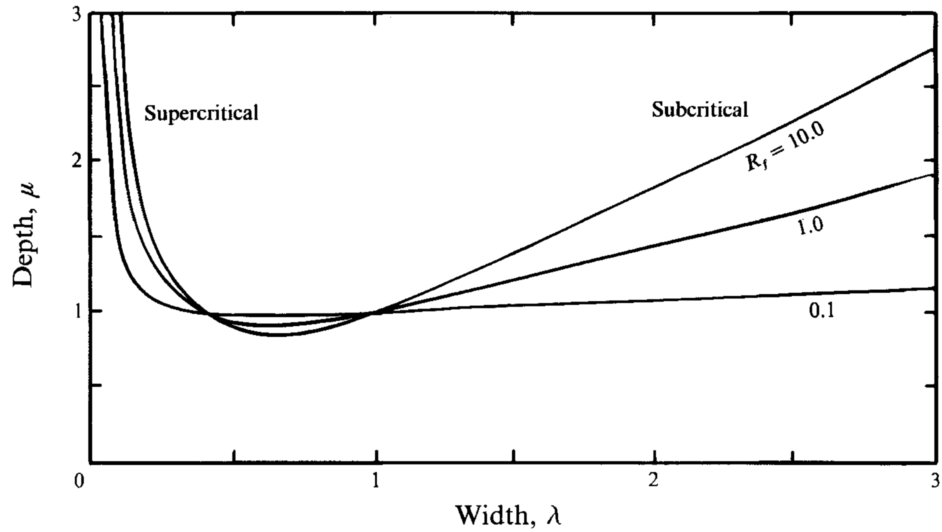

FIGURE 3. The dependence of the dimensionless current width λ upon the dimensionless depth μ for $R_\beta = 0.1$ and $R_f = 0.1$, 1 and 10.

at which point

$$\lambda = \tfrac{2}{3}[\mu(1+R_\beta)+R_f(\mu-1)]. \tag{3.17}$$

These flow profiles on a midlatitude β-plane include a region of recirculation if

$$1 \leqslant \tfrac{2}{3}R_f\left(\frac{1-\mu}{\mu-\lambda}\right) \leqslant 2 \tag{3.18}$$

4. A simple model of planetary eddies

We now discuss the evolution of the current downstream of the topography. We focus upon the self-similar equatorial flows of §3 as these are analytically tractable and therefore rather illuminating.

If the flow changes smoothly from the subcritical to the supercritical solution branch on passing over the topography, then we may expect that at some point downstream of the flow, the flow may revert to the subcritical solution branch; this transition may be effected through the generation of planetary eddies or planetary waves. As the current adjusts back from the supercritical to the subcritical solution branch, the momentum and mass flux of the current is conserved; however, energy is dissipated. Assuming for convenience that the current has the same profile (3.8) upstream and downstream of the jump, conservation of momentum flux enables us to calculate the final subcritical flow. We can then calculate the energy dissipation in this transition and use this to infer the strength and dissipation rate of eddies or planetary waves which effect the change in the solution branch of the current.

The zonal momentum flux, M, is defined to be

$$M = \mu H_0 \int_0^L \mathrm{d}y \left(u^2+P-\frac{f_0+\beta y}{\mu(x)H_0}\psi\right). \tag{4.1}$$

It follows from (2.1a) and the continuity equation (2.2) that in a region of constant depth, the momentum flux is conserved, $\mathrm{d}M/\mathrm{d}x = 0$ because no forces act on the flow. If the current adjusts from the supercritical to the subcritical solution branch in a region of constant depth downstream then we can use (4.1) to calculate the width of the subcritical flow, given the supercritical flow, assuming for simplicity that the flow retains the same velocity profile.

We may integrate (2.1*b*) to derive an expression for the variation of pressure across the parallel zonal flow,

$$P(y) = P(L) - \int_L^y \mathrm{d}y \frac{\beta\psi}{H(x)} - \left[\frac{\beta y \psi}{H(x)}\right]_y^L + \frac{f_0(\psi(y)-\psi(L))}{H(x)}. \tag{4.2}$$

Combining (3.2) for u with (4.1) and (4.2), it may be shown that

$$M = \frac{1}{H_0 \mu\lambda} \int_0^L [\Psi_y^2 + \beta y \lambda^3 \mu \Psi H_0] \,\mathrm{d}y + \frac{\lambda^2 \beta L^2 \Phi}{2}, \tag{4.3}$$

where λ represents the width of the current relative to its width upstream of the topography, μ represents the ratio of the depth at the present location relative to that far upstream, and m is a constant which we can set to zero without loss of generality.

Substituting in our calculated value for the shape of the current, as given by (3.8) with $R_f = 0$, we obtain the expression

$$M = \frac{2L^5 \beta^2 R_\beta^2 H_0}{15\mu\lambda}\left(1+\frac{\lambda^3\mu}{4R_\beta}\right) = \frac{M_0}{\mu\lambda}\left(1+\frac{\lambda^3\mu}{4R_\beta}\right). \tag{4.4}$$

This represents the momentum flux at some point beyond the topography where the relative depth is μ and current width is λ. In figure 4, we have plotted M/M_0 as a function of λ for $\mu = 1$ on an equatorial β-plane. Note that in this class of solutions, for each value of μ the momentum flux associated with the current is bounded below, $M \geqslant 3M_0/(16R_\beta \mu^2)^{\frac{1}{3}}$, and this is attained when $\lambda = (2R_\beta/\mu)^{\frac{1}{3}}$. It may be seen that if the flow downstream of the topography, described in §3, has width λ which satisfies $\lambda < (2R_\beta/\mu)^{\frac{1}{3}}$, then there is a second flow solution downstream, which has the same velocity profile, mass and momentum flux but which is wider.

The energy flux is defined by

$$G = \mu H_0 \int_0^L u\left(\frac{u^2}{2}+P\right)\mathrm{d}y \tag{4.5}$$

and using equations for the pressure, (4.2), and velocity, (3.2), expression (4.5) may be simplified to the form

$$G = \frac{1}{H_0^2\lambda^2\mu^2}\int_0^L -\left[\Psi_y^3 + \frac{H_0\beta\lambda^3\mu}{2}\Psi^2\right]\mathrm{d}y + \frac{\beta\lambda L \Phi^2 H_0}{2} + c, \tag{4.6}$$

where c is a constant which we set to zero without loss of generality. Using the self-similar solution (3.8) this simplifies to

$$G = \frac{\beta^3 L^7 R_\beta^3 H_0}{35\lambda^2\mu^2}\left(1+\frac{\lambda^3\mu}{R_\beta}\right) = \frac{G_0}{\lambda^2\mu^2}\left(1+\frac{\lambda^3\mu}{R_\beta}\right). \tag{4.7}$$

In figure 4, we have also plotted the variation of the energy flux, G/G_0, as a function

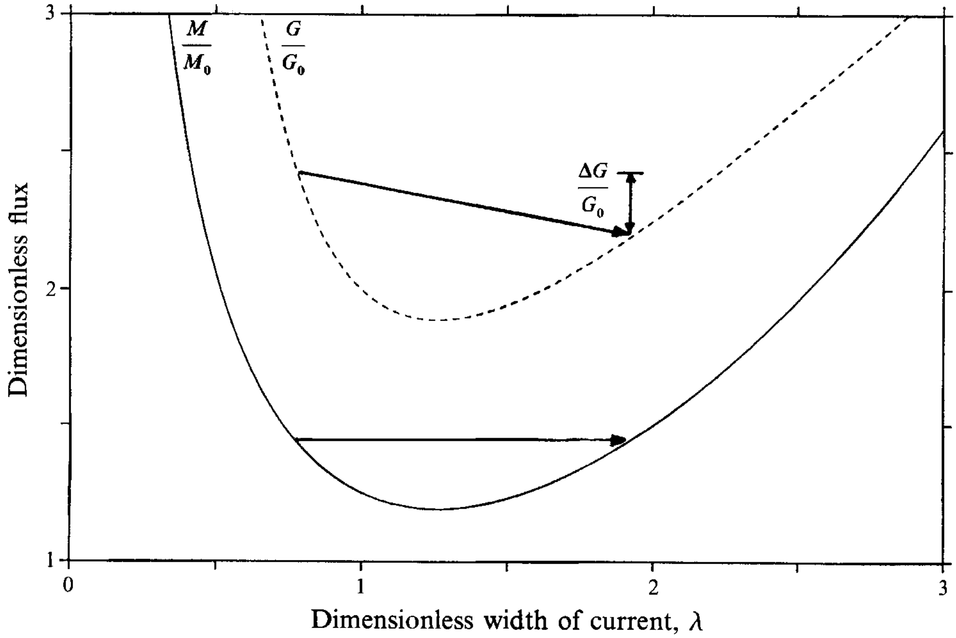

FIGURE 4. The momentum flux (solid) and the energy flux (dashed) as a function of the relative width of the current, λ. The arrows represent the path followed across a transition from the supercritical solution branch to the subcritical branch. Here, $R_\beta = 1$, $R_f = 0$ (equatorial) and $\mu = 1$.

of λ with $\mu = 1$. One may deduce from (4.7) that the energy flux associated with this class of flows is bounded below by $E \geqslant 3G_0/(2R_\beta \mu^2)^{\frac{2}{3}}$, and that the minimum is attained when $\lambda = (2R_\beta/\mu)^{\frac{1}{3}}$.

It may be seen from figure 4 that at a given depth and for a given momentum flux, the supercritical narrower current has a larger energy flux. Indeed, the difference in the energy flux, ΔG, between the narrow current of width $\lambda_1 L$ and the wider current of width $\lambda_2 L$ is given from (4.7) by

$$\Delta G = \frac{G_0}{\mu^2 \lambda_1^2 \lambda_2^2}(\lambda_2 - \lambda_1)\left((\lambda_1 + \lambda_2) - \frac{\mu \lambda_1^2 \lambda_2^2}{R_\beta}\right). \tag{4.8}$$

Using (4.4) it may be shown that (4.8) is always positive when $\lambda_2 > \lambda_1$. In figure 5 we have plotted M/M_0 as a function of G/G_0 for the parameters of figure 4, $R_\beta = 1$ and $\mu = 1$; $\Delta G > 0$ when the current jumps from the supercritical narrow solution branch to the subcritical wide solution branch and ΔG is typically of order 1.

We have now established that downstream of the topography, if the current has width $\lambda < (2R_\beta/\mu)^{\frac{1}{3}}$, then there is a second wider flow solution with the same velocity profile and momentum flux but a lower energy flux. We therefore suggest that the narrower flow may become unstable and the flow solution may jump to the wider flow of lower energy. In figure 5, the arrow shows an example of a transition from the narrower solution branch (λ_{sup}) to the wider solution branch (λ_{sub}); both flows have the same momentum flux and velocity profile, but the wider flow has a lower energy flux. This transition from a high energy state to a lower energy state is analogous to a hydraulic jump, and may arise due to nonlinear Rossby waves propagating westward in an analogous fashion to nonlinear gravity waves in a stratified flow; the narrower flow upstream of the jump is supercritical and the wider flow downstream

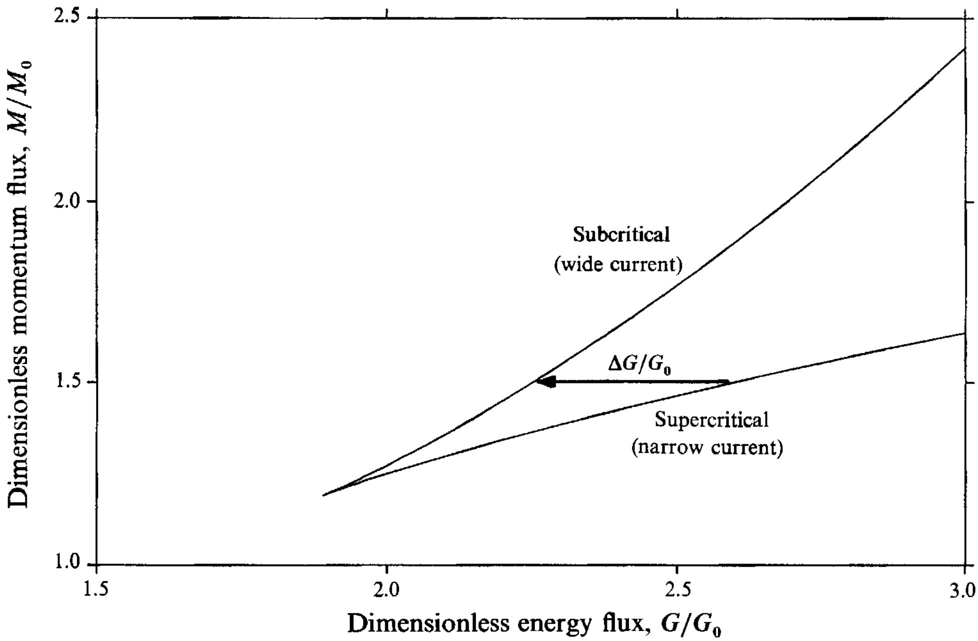

FIGURE 5. The variation of the energy flux with the momentum flux using the parameters of figure 4.

is subcritical. We should mention that there is no reason why the flow downstream of the transition should be of the same shape as that upstream; however, these self-similar flows are analytically tractable and therefore provide a simple but powerful means of describing the phenomenon.

If much of the energy is dissipated locally through the generation of intense eddies (or local transient waves), the present calculation allows us to estimate the order of magnitude of the dissipation rate. If we assume that the eddy has dynamic vorticity Q then it will dissipate energy at a rate νQ^2, where ν represents the viscosity. We may therefore write down the simple energy balance

$$L^2 H_0 Q^2 \nu \sim \Delta G \tag{4.9}$$

and so the vorticity of the eddy scales approximately as

$$Q \sim \left(\frac{\beta^3 L^5 R_\beta^3}{35\nu}\right)^{\frac{1}{2}} \left(\frac{\Delta G}{G_0}\right)^{\frac{1}{2}}. \tag{4.10}$$

We may interpret the scaling (4.10) for the vorticity associated with the planetary eddies in a more general fashion. Depending upon the mechanism which produces the supercritical flow and the flow solution which obtains downstream, the numerical factor $\Delta G/G_0$ will change. However, assuming that the hydraulic jump results from essentially inertial processes, the scaling (4.10) should be independent of the mechanism of generation of the eddies.

5. Flow passing across a coastline bump

The method we introduced for calculating the flow across a region of topography in §3 can also be applied to the problem of flow along an east–west boundary of variable latitude, for example, as occurs in the circum-polar current (figure 6). In this situation, we assume for simplicity that the depth of the fluid is fixed and so $\mu = 1$; it follows from the discussion of §3, that the effect of the background rotation, f_0, is zero. If the current flows along a boundary whose latitude changes smoothly from $y = 0$ to $y = b$ then the Bernoulli function and hence the potential vorticity are conserved on each streamline. We now derive a self-similar solution for the flow upstream and downstream of the bump. The problem is equivalent to the flow of a stratified fluid over a bump in a channel of fixed width, as first considered by Long (1955); however, a self-similar class of solutions to the stratified problem are not known to the author and, following the analogy in the Appendix, the results below may be modified for that context.

We choose $\Psi(y)$ to be the upstream streamfunction, and $\psi(z)$ be the downstream streamfunction, where $z = \lambda(y+b)$, and seek a similarity solution such that

$$\lambda\psi_z(z(y)) = \Psi_y(y). \tag{5.1}$$

Using the notation of §3, and applying the conservation of potential vorticity on streamlines yields the equation

$$\Psi_{yy} = \frac{H_0\beta\lambda^2}{1+\lambda}\left(y - \frac{\lambda b}{1-\lambda}\right) \tag{5.2}$$

which is analogous to (3.7). Equation (5.2) has a solution of the form

$$\Psi = \frac{H_0\beta\lambda^2}{1+\lambda}\left[\frac{y^3}{6} - \frac{By^2}{2} + \left(BL - \frac{L^2}{2}\right)y\right], \tag{5.3}$$

where $\Psi_y(L) = 0$ and $B = \lambda b/(1-\lambda)$. In order that the total transport of the current is conserved, $\Psi(L) = -H_0\Phi$. Therefore, λ must satisfy the cubic relationship

$$\lambda^2(1+R_\beta - \lambda(1+3b/(2L))) = R_\beta, \tag{5.4}$$

where $R_\beta = 3\Phi/(\beta L^3)$ is the Rossby number for the flow as defined in §3. If $b = 0$ downstream then the flows outside the region $b \neq 0$ become identical to those of §3 in which the depth returned to its original value downstream. The boundary perturbation may thus cause the flow to change solution branch: two flows are possible downstream, the original flow and a narrower supercritical flow.

If $1+3b/2L > 0$, then the cubic for λ, (5.4), has two positive roots if

$$4(1+R_\beta)^3 > 27R_\beta(1+3b/2L)^2 \tag{5.5}$$

corresponding to two different self-similar solution branches for the current. If $b \neq 0$ downstream then each solution branch has a different velocity profile; on each solution branch, the flow upstream and downstream of the boundary perturbation have the same profile but different widths. The solution branches coincide when (5.5) becomes an equality. At this point, the current width downstream is given by

$$\lambda = \frac{2(1+R_\beta)}{3(1+3b/2L)}. \tag{5.6}$$

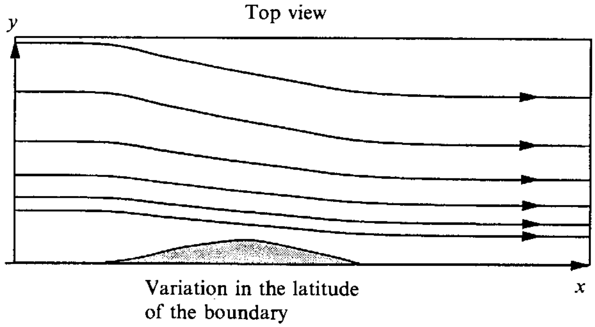

FIGURE 6. Schematic of the flow past a region where the boundary latitude changes.

Note that the extrema of the streamfunction upstream of the bump, (5.5), occur at L and $2B-L$. Therefore, if $\frac{1}{2}L < B < L$, the solution includes a region of recirculation, while if $B > L$, the flow is unidirectional. From (5.6) we note that $B = L$ when

$$b = 2(2R_\beta - 1)/(5-4R_\beta) \tag{5.7}$$

6. Geophysical relevance

The existence of multiple solutions of large-scale zonal, planetary motions over topography has been a subject of significant interest in the atmospheric sciences. Two pioneering studies were those of Charney & DeVore (1979) and Hart (1979) who considered the problem of a barotropic, zonal flow over topography of relatively small amplitude, including a fractional forcing term. By using a truncated Fourier representation of the flow field (Charney & DeVore) and a small-amplitude expansion of the equations of motion (Hart) they discovered that multiple steady solutions may exist. In contrast, the inviscid solutions that we presented in §3 are unforced and are valid for topography of finite amplitude.

Following the spirit of these studies, and the earlier work of Ball (1954) we suggest that the present model may be useful in understanding some aspects of atmospheric flow over mountains, although we are ignoring the baroclinic component of the flow; in particular Hart (1979) mentioned the role of the Rocky Mountains in controlling atmospheric winds. However, we mention that the jetstream is more complex than flows described by our inertial model, and so other processes may dominate the effect we are presently considering. The energy injected into eddies across the transition from the super- to subcritical flows downstream of the topography, as estimated in §4, may be useful in understanding the energy input to mixing in the stratosphere.

Charney & Flierl (1981) noted that the Kuroshio current, south of Japan, exhibits bimodal behaviour and presented a simple barotropic model, arguing that the bimodality arose from variations in the latitude of the coastline. However, our new hydraulic theory may elucidate some aspects of the behaviour of model western boundary currents immediately following separation from the coast, where they appear to behave as inertial, eastward-propagating zonal currents. Although actual boundary currents have an important baroclinic ingredient to their dynamics, the transition from a relatively rapid and confined current, to a wider and somewhat slower flow offshore may be a manifestation of the transition from supercritical to subcritical flow as discussed in §§3 and 5; this interpretation is consistent with the generation of intense eddies just after boundary separation which are observed for example, beside the East Australian current (Boland & Church 1981) and in numerical simulations of the wind-driven ocean circulation (Bryan 1963; Pantaleev 1985).

Extension of the present theory to incorporate some of the crucial effects which result from the baroclinicity, including the surfacing of the lower layer and the interlayer gravitational coupling (Ou & DeReuter 1987; Stommel & Luyten 1987) is essential in order to apply the model to the Gulf Stream or the Agulhas current and this is presently under study. However, we mention here that the idea that topography exerts a strong control upon the path of the Gulf Stream was suggested some time ago (Warren 1963).

7. Conclusions

We have developed a control theory for eastward-propagating zonal currents on a β-plane flowing over topography which varies slowly with latitude. We have found two classes of solutions which have the same velocity profile upstream and downstream of a region of topography. In one of these classes of solution, if the depth downstream differs from that upstream the current width increases downstream, and in the other class the current width decreases. In the special case of an equatorial β-plane both of the solution branches have the same velocity profile while, in general, at other latitudes each class of solution for the current has a different current profile. The narrow rapidly propagating flow is supercritical and the wider slow flow is subcritical. The topography can therefore cause the current to change continuously from being subcritical upstream to supercritical downstream. In the case of an equatorial flow we can therefore follow the current as it moves over the topography from the subcritical solution, through a control, and onto the supercritical solution. We calculate that the critical height of the topography, for a given flow rate, which is necessary to control the flow, is $H_0[27R_\beta/(4(1+R_\beta)^3)]^{\frac{1}{4}}$.

We show that on an equatorial β-plane, downstream of the topography, the supercritical narrow flow is of higher energy than the corresponding subcritical flow which has the same momentum flux. Therefore, we suggest that a transition in the flow may occur downstream of the topography across which the flow returns to the subcritical solution branch with the same momentum flux. This transition may manifest itself as intense planetary eddies or planetary waves. We calculate the energy dissipated across such a transition and thereby purpose a simple scaling of the eddy strength, on an equatorial β-plane, assuming the dissipation is effected by intense planetary eddies.

We have presented some further similarity solutions which describe the flow of a zonal boundary current along a coastline whose latitude changes slowly from one value upstream to a second value downstream. We show that again two families of self-similar solutions exist in this problem; these solutions are similar to those resulting at midlatitudes from the variation of the topography with latitude.

In the Appendix, we extend the analogy between stratified flow and motion on a β-plane, which was proposed by Ball (1954). We show that steady, inviscid motion on a β-plane of fixed depth is identical to the class of steady horizontal stratified flows along a channel of fixed width in which the density is linearly related to the streamfunction. In the special case of an equatorial β-plane, variations in the depth of the β-plane flow correspond to variations in the width of the channel in the stratified flow problem. We have thereby shown that our theory is equivalent to that of Benjamin (1981), who considered the motion of a stratified flow in a channel with a constriction. Recognition of the exact analogy between zonal planetary-scale flows and non-rotating stratified channel flow clearly identifies some of the limitations associated with the simpler hydraulic control models proposed by Rossby (1949) and later by Armi (1989), which were based upon a similarity with the hydraulics of non-rotating homogenous channel flow.

I started this work at the Scripps Institution of Oceanography funded by the Green Foundation of the Institute of Geophysics and Planetary Physics as a Green Scholar and the NSF. I have had useful discussion with Larry Armi, Glenn Ierley and Bill Young about this and related problems.

Appendix. The analogy between zonal planetary-scale flow and stratified channel flow

Ball (1954) showed that the vorticity equations for the β-plane and for a stratified flow are very similar; he then deduced that under certain circumstances, the dynamics of the two systems are analogous. We extend and develop this result by showing that steady zonal flows on the β-plane are identical to a subclass of flows of a stratified fluid in which the density varies linearly with streamfunction.

The steady equations of motion of a non-diffusing, incompressible stratified fluid are

$$\rho(\boldsymbol{u}\cdot\boldsymbol{\nabla})\,\boldsymbol{u}+g\rho\hat{\boldsymbol{y}} = -\boldsymbol{\nabla}\Pi \tag{A 1}$$

and

$$(\boldsymbol{u}\cdot\boldsymbol{\nabla})\,\rho = 0, \tag{A 2}$$

where Π is the fluid pressure and $\rho(y)$ the fluid density. Equation (A 1) may be rewritten as

$$\rho(\boldsymbol{u}\cdot\boldsymbol{\nabla})\,\boldsymbol{u}-gy\boldsymbol{\nabla}\rho = -\boldsymbol{\nabla}(\Pi+gy\rho). \tag{A 3}$$

Equation (A 2) implies that ρ is constant along streamlines and so one class of solutions is

$$\boldsymbol{\nabla}\rho = F(\psi)(-v,u). \tag{A 4}$$

Therefore (A 3) may be simplified to the form

$$\rho(\boldsymbol{u}\cdot\boldsymbol{\nabla})\,\boldsymbol{u}-gyF(\psi)(-v,u) = -\boldsymbol{\nabla}(\Pi+gy\rho). \tag{A 5}$$

By taking the vector product of $\boldsymbol{u}$ with (A 5) and combining this with (A 2), one may show that the quantity $\frac{1}{2}\rho u^2+\Pi+gy\rho$ is conserved following the flow; this is the Bernoulli constant. In a Boussinesq fluid, in which $\rho = \rho_0+\rho_1$ with $\rho_0 \gg \rho_1$ where ρ_0 is a constant, (A 5) reduces to

$$(\boldsymbol{u}\cdot\boldsymbol{\nabla})\,\boldsymbol{u}-gy\boldsymbol{\nabla}\hat{\rho}_1 = -\boldsymbol{\nabla}(\hat{\Pi}+gy(1+\hat{\rho}_1)) \tag{A 6}$$

and the Bernoulli constant becomes $\frac{1}{2}u^2+\hat{\Pi}+gy(1+\hat{\rho}_1)$ where $\hat{\Pi} = P/r_0$ and $\hat{\rho}_1 = \rho_1/\rho_0$.

A.1. An equatorial *β-plane*, $f_0 = 0$

Equation (A 6) is very similar to (2.3). The equations are identical when $f_0 = 0$ and

$$\boldsymbol{\nabla}\hat{\rho}_1 = l(v,\,-u), \tag{A 7}$$

where λ is a constant, so that (A 6) becomes

$$(\boldsymbol{u}\cdot\nabla)\,\boldsymbol{u}-gy\lambda(v,\,-u) = -\nabla(\hat{\Pi}+gy(1+\hat{\rho}_1)). \tag{A 8}$$

This result may be simply stated:

I: the two-dimensional inviscid steady flows on a barotropic, unforced, equatorial β-plane are identical to the steady, two-dimensional flows of a stratified, inviscid Boussinesq fluid in which $\nabla\rho = \lambda(v,\,-u)$, where the pressure, P/ρ_0, on the β-plane is equivalent to the sum of the pressure, Π/ρ_0, and the potential energy $gy\rho/\rho_0$ in the stratified fluid.

In regions of constant depth, if the density of the stratified fluid is linearly dependent upon the velocity streamfunction, then the streamlines of the flow in the stratified fluid are identical to those of the analogous flow on the β-plane. Through this analogy, we may associate a density with each streamline in a planetary-scale flow.

A.2. *Non-equatorial β-plane of constant depth*

In the different case of a non-equatorial β-plane of constant depth H we are able to introduce a velocity streamfunction $\hat{\psi} = \psi/H$. In a stratified fluid in a channel of constant width we can introduce a velocity streamfunction, ϕ, and it then follows from (A 6) that $\rho = G(\phi)$ for some function of ϕ. If we set (cf. (A 7))

$$\rho = \rho_0(1+\lambda\phi), \tag{A 9}$$

where λ is a constant then we can rewrite the stratified flow equation (A 6) in the form

$$(\boldsymbol{u}\cdot\nabla)\,\boldsymbol{u}-gy\lambda\nabla\phi = -\nabla(\Pi/\rho_0+gy(1+\lambda\phi)). \tag{A 10}$$

This is identical to the equation of motion on the β-plane (2.3) and yields our second identity, that

II: the two-dimensional inviscid steady flows on a barotropic β-plane of constant depth, H, and with transport streamfunction ψ are identical to the two-dimensional, steady flows of a stratified, inviscid Boussinesq fluid, with velocity streamfunction ϕ in a channel of fixed width in which $\rho = \rho_0(1+\lambda\phi)$. The sum of the pressure, P/ρ_0, and the reference Coriolis pressure term $-f_0\,\psi/H$ on the β-plane are identical to the sum of the pressure Π/ρ_0 and the potential energy of the fluid $gy\rho_0(1+\lambda\phi)$ in the stratified fluid.

This equivalence between the two systems implies that the term $\beta y\nabla\psi/H$ in (2.3) is analogous to the term $gy\nabla\rho$ in (A 3). Therefore, the variation of the Coriolis force with latitude may be interpreted as exerting a force upon variations of the transport ψ in an identical fashion to the force exerted by gravity upon density differences ρ. Using (A 2) we can identify a density with each streamline and thereby interpret the motion on a β-plane as a special class of motions in a stratified fluid. This observation allows a simple and intuitive interpretation of the dynamical balances operating in many steady flows on the β-plane. In a stratified fluid, density differences exert a buoyancy force upon the flow and these may be viewed as being decoupled from the motion; on the β-plane we have shown that it is the variation of the streamfunction of the fluid which determines the 'buoyancy force' upon the flow, but the streamfunction is implicitly linked to the flow. By associating a density, proportional to the streamfunction, to any flow solution on the β-plane, we may then interpret the flow solution as that of a stratified fluid.

A.3. *Non-equatorial β-plane of variable depth*

In the more general situation, the Coriolis force associated with the background rotation, f_0/H varies with depth and we cannot incorporate this effect purely as an additional contribution to the pressure field (equation (2.3)). Therefore there is no exact analogy with the class of simple stratified channel flows described by (A 5).

It is of interest to note that in the shallow-water approximation for the stratified flow the hydrostatic pressure is (Benjamin 1981)

$$\Pi(y) = -\int_L^y \mathrm{d}y\, g\rho + \Pi(L) = \Pi(L) + \int_L^y \mathrm{d}y\, gy\rho_y + [gy\rho]_y^L, \tag{A 11}$$

whereas in the rotating flow, the pressure is given by

$$P(y) = P(L) - \int_L^y \mathrm{d}y\, \beta\psi/H(x) - [\beta y\psi/H(x)]_y^L + f_0(\psi(y) - \psi(L))/H(x). \tag{A 12}$$

These expressions identify the equivalence between the quantity $P - f_0\,\psi(y)/H$ on the β-plane and the sum of the hydrostatic pressure, Π, plus the potential energy $gy\rho$ in a stratified fluid. However, it also gives the converse result that the β-plane quantity $P(y) - (f_0\,\psi + \beta y\psi)/H$ is equivalent to the pressure in a stratified fluid Π.

A.4. *The analogy and previous hydraulic theories*

Two interesting similarities between zonal planetary-scale flows and stratified channel flows may be noted using the analogy described above: (i) the effect of variations of the depth of the β-plane upon a zonal planetary-scale flow are analogous to variations in the width of the channel upon a stratified channel flow, and (ii) the effect of variations in the latitudinal position of the zonal boundary upon a zonal planetary-scale flow are analogous to variations in the depth upon a stratified channel flow. It is recognition of this analogy, and the subsequent adaption and application of techniques developed for stratified flow hydraulics (Benjamin 1981) that distinguishes this work from earlier models of the hydraulics of planetary-scale flows.

As mentioned in the introduction, Armi (1989) developed a somewhat different model of the hydraulic control of zonal currents on a β-plane, following the work of Rossby (1949). The models developed in these earlier works were derived from the qualitative similarity with the hydraulics of a homogeneous, non-rotating channel flow. Rossby (1949) and Armi (1989) calculated the energy flux G of a zonal current (equation (2.4)). As mentioned in §2, this represents the integral of the Bernoulli function across the flow. Both Rossby and Armi derived expressions for G in terms of an unknown shape functions for the flow and a non-dimensional number, $R_{0\beta} = u_{\mathrm{m}}/(\alpha'\beta a^2)$, described as the Froude/Rossby number, where α' is a shape function, u_{m} the maximum velocity in the flow and a the width of the flow. By assuming that the shape functions of the current do not change alongstream, both Rossby and Armi showed that the energy flux has a minimum value when $R_{0\beta} = 1$. Armi interpreted this as the critical flow, with wider slower currents being subsonic and narrower faster currents being supersonic.

By assuming that, in the absence of external forces, the energy flux remains fixed, that is $\mathrm{d}G/\mathrm{d}x = 0$, Armi then argued that as the background pressure field changes, the Rossby/Froude number of the flow changes and the flow gradually becomes supersonic. As an example, he considered a source–sink flow in which he proposed that the background pressure field would change near to the sink, and the flow could become supersonic. However, he did not give any quantitative details of the change in pressure field due to the sink, but simply showed that as the term representing the contribution of the background pressure to the energy flux, G, changed the flow could become supersonic. However, in order to use the theory one must assume that the flow has a particular shape function. Armi calculated the shape functions associated with a number of arbitrary, but simple, flow fields; these flow fields were not constrained to conserve potential vorticity along individual streamlines.

A major advance of the present theory, described in §3, is the calculation of exact solutions for the streamfunction, and the demonstration that either topography or a variation in the boundary latitude may exert an explicit control upon the flow. The methodology we have developed is motivated by the analogy with continuously stratified channel flow described above; in such problems the Bernoulli function must be conserved on each streamline (Benjamin 1981) – although in fact in §3 we used the conservation of potential vorticity on each streamline, it is possible, though somewhat more detailed, to rederive all our results for zonal planetary-scale flows using the conservation of the Bernoulli function (§2) on each streamline. In the simpler problem in which there are one or two layers of fluid of distinct densities, a uniform flow may develop in each layer; conservation of the energy flux integrated across the flow may then be used to calculate the flow, since the 'shape factors' are trivial. However, when the flow is not uniform, as in the present work, the conservation of potential vorticity or equivalently the Bernoulli function must be used to determine the shape of the flow.

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